B.Tech First Year: Regular Course Lecture Plan Session 2022-23

Subject Name	Engineering Mathematics-I (BAS-103)

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Signature	
Name of Subject Head	Dr. Ruchi Garg

UNIT-V (VECTOR CALCULUS)

Scalar Point function:→ If to each point P(x,y,z) o
region R in space there corresponds a unique scal
then f is called a scalar point function.

Vector Point Function:→ If to each point P(x,y,z) of a r
in space there corresponds a unique vector f(P), then f is
vector point function.

Vector Differential Operator Del i.e. ∇ :→ (20
It is defined as

$$\boxed{\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$

Gradient of a Scalar Function:→ If $\phi(x,y,z)$ be a scalar
then $\boxed{\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}}$ is called the gradient of the scalar function ϕ. (20

It is denoted by grad ϕ and $\boxed{\text{grad}\,\phi = \nabla\phi}$.

Note:→ The coplanarity of three vectors is defined when the triple scalar product is zero. i.e. If $\vec{u}, \vec{v}$ and $\vec{w}$ a
three vectors then they are coplaner when $\vec{u}\cdot(\vec{v}\times$

Q.1 → Find a unit vector normal to the surface $x^3+y^3+3xyz=3$
the point (1,2,-1). (2013-

Sol :→ Let $\phi = x^3+y^3+3xyz-3$, then $\frac{\partial\phi}{\partial x} = 3x^2+3yz$, $\frac{\partial\phi}{\partial y} = 3y^2+3$
$\frac{\partial\phi}{\partial z} = 3xy$

$\therefore \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = (3x^2+3yz)\hat{i} + (3y^2+3xz)\hat{j} + 3xy$

At $(1,2,-1)$, $\nabla\phi = -3\hat{i}+9\hat{j}+6\hat{k}$ which is a vector normal to the given surface at $(1,2,-1)$. Hence a unit vector normal to the given surface at $(1,\ ,-1)$

$$= \frac{\nabla\phi}{|\nabla\phi|} = \frac{-3\hat{i}+9\hat{j}+6\hat{k}}{\sqrt{(-3)^2+(9)^2+(6)^2}} = \frac{-3\hat{i}+9\hat{j}+6\hat{k}}{3\sqrt{14}} = \frac{-\hat{i}\ 3\hat{j}+2\hat{k}}{\sqrt{14}}$$

Q.2:- If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = yz+zx+xy$. Prove that grad u, grad v and grad w are coplanar. (2014-5)

Sol^n:-

$$\text{grad}\,u = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x+y+z)$$

$$= \hat{i}+\hat{j}+\hat{k}$$

$$\text{grad}\,v = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$$

$$= 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$\text{grad}\,w = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(yz+zx+xy)$$

$$= (y+z)\hat{i}+(z+x)\hat{j}+(x+y)\hat{k}$$

For vectors to be coplanar, their scalar triple product is zero.

$$\text{grad}\,u\cdot(\text{grad}\,v \times \text{grad}\,w) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ z+y & z+x & y+x \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ z+y & z+x & y+x \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ z+y & z+x & y+x \end{vmatrix} \quad R_2 = R_2+R_3$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ z+y & z+x & y+x \end{vmatrix} = 0$$

Q.3:- For the scalar field $u = \frac{x^2}{2}+\frac{y^2}{3}$, find the magnitude of gradient at the point $(1,3)$. (2016-17)

Sol^n:-

$$\text{grad}\,u = \nabla u = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}\right)\left(\frac{x^2}{2}+\frac{y^2}{3}\right)$$

$$= x\hat{i}+\frac{2y}{3}\hat{j}$$

At $(1,3)$ $\text{grad}\,u = \hat{i}+2\hat{j}$

Now $|\text{grad}\,u|_{at\,(1,3)} = \sqrt{(1)^2+(2)^2} = \sqrt{5}$

Q.4:- Define del ∇ operator and gradient. (2018-19)

Sol^n:- Refer the above given definition.

Q.5:- If $\phi = 3x^2y - y^3z^2$, find grad ϕ at point $(2,0,-2)$. (2018-19)

Sol^n:-

$$\text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(3x^2y-y^3z^2)$$

$$= (6xy)\hat{i}+(3x^2-3y^2z^2)\hat{j}+(-2y^3z)\hat{k}$$

At point $(2,0-2)$, grad $\phi = 12\hat{j}$

Q.6:- Find grad ϕ at the point $(2,1,3)$ where $\phi = x^2+yz$. (2019-20)

Sol^n:- Proceed as Q.5.

Q.7:- Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$.

Sol^n:- Let $\phi_1 = x^2+y^2+z^2 = 9$ and $\phi_2 = x^2+y^2-z = 3$

Then $\text{grad}\,\phi_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k}$ and $\text{grad}\,\phi_2 = 2x\hat{i}+2y\hat{j}-\hat{k}$

Let $\vec{n_1} = \text{grad}\,\phi_1$ at the point $(2,-1,2)$ and $\vec{n_2} = \text{grad}\,\phi_2$ at the point $(2,-1,2)$ then

$\vec{n_1} = 4\hat{i} - 2\hat{j} + 4\hat{k}$ and $\vec{n_2} = 4\hat{i} - 2\hat{j} - \hat{k}$

The vectors $\vec{n_1}$ and $\vec{n_2}$ are along normals to the two surfaces at the point $(2,-1,2)$. If θ is the angle between these vectors, then

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{4(4)-2(-2)+4(-1)}{\sqrt{16+4+16}\,\sqrt{16+4+1}} = \frac{16}{6\sqrt{21}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

Properties of Gradient

(a) If ϕ is a constant scalar point function, then $\nabla\phi = \vec{0}$

(b) If ϕ_1 and ϕ_2 are two scalar point functions, then

(i) $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$

(ii) $\nabla(c_1\phi_1 + c_2\phi_2) = c_1\nabla\phi_1 + c_2\nabla\phi_2$ where c_1, c_2 are constant

(iii) $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$

(iv) $\nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2^2}$, $\phi_2 \neq 0$

Notes: (1) The gradient of a scalar field ϕ is a vector normal to the surface $\phi = c$ and has a magnitude equal to the rate of change of ϕ along the normal.

(2) $\boxed{\text{grad} f = \frac{\partial}{\partial r}(f)\cdot\hat{r}}$ gradient in polar coordinate

Ex: Prove that $\text{grad}\, r^n = n r^{n-2}\vec{r}$ where $r = |\vec{r}|$

Sol: $\text{grad}\, r^n = \frac{\partial}{\partial r}(r^n)\cdot\hat{r} = n r^{n-1}\hat{r} = n r^{n-2}\vec{r}$

where $\hat{r}$ is the unit vector in the direction of $\vec{r}$.

Directional Derivative: Let $f(x,y,z)$ be a scalar va

directional derivative of $f(x,y,z)$ at

B in the direction of a vector $\vec{e}$ is given by

$\boxed{\text{Directional derivative} = (\nabla f)_{\text{at } \vec{B}} \cdot \hat{e}}$ where $\hat{e}$ unit vecto vector $\vec{e}$

Q: Find the directional derivative of v^2, where $\vec{v} = xy^2\hat{i}$ at the point $(2,0,3)$ in the direction of the outward to the sphere $x^2+y^2+z^2 = 14$ at the point $(3,2,1)$.

Sol: $\vec{v} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ then $v^2 = x^2y^4 + z^2y^4 + x^2$

Now

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^4 + z^2y^4 + x^2z^4)$$

$$= \hat{i}(2xy^4 + 2xz^4) + \hat{j}(4x^2y^3 + 4z^2y^3) + \hat{k}(2zy^4$$

At point $(2,0,3)$, $\nabla f = 324\hat{i} + 432\hat{k}$

Normal to the sphere $x^2+y^2+z^2 = 14 \equiv \phi$ is

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-14) = 2x\hat{i} + 2y$$

At $(3,2,1)$ $\nabla\phi = 6\hat{i} + 4\hat{j} + 2\hat{k}$

If $\hat{n}$ is a unit vector in outward normal to the

the $\hat{n} = \frac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{36+16+4}} = \frac{1}{\sqrt{56}}(6\hat{i}+4\hat{j}+2\hat{k})$

$\therefore$ Directional derivative of f in the outward normal to sphere $= \nabla f\cdot\hat{n}$

$$= (324\hat{i} + 432\hat{k})\frac{1}{\sqrt{56}}(6\hat{i}+4\hat{j}+2\hat{k}) = \frac{1944+}{\sqrt{56}}$$

Q.2:- Find the directional derivative of $\phi = (x^2+y^2+z^2)^{-1/2}$ at the point (3,1,2) in the direction of the vector $yz\hat{i}, xy\hat{j} + y\hat{k}$. (2013-14)

Solⁿ:- $\phi = (x^2+y^2+z^2)^{-1/2}$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-1/2}$$

$$= \hat{i}\left[\frac{-1}{2}(x^2+y^2+z^2)^{-3/2}(2x)\right] + \hat{j}\left\{\frac{-1}{2}(x^2+y^2+z^2)^{-3/2}(2\ )\right\} + \hat{k}\left\{\frac{-1}{2}(x^2+y^2+z^2)^{-3/2}(2z)\right\}$$

$$= -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{3/2}} = -\frac{3\hat{i}+\hat{j}+2\hat{k}}{14\sqrt{14}} \text{ at } (3,1,\ )$$

Let $\hat{a}$ be the unit vector in the given direction, the

$$\hat{a} = \frac{yz\hat{i}+2x\hat{j}+xy\hat{k}}{\sqrt{y^2z^2+z^2x^2+x^2y^2}} = \frac{2\hat{i}+6\hat{j}+3\hat{k}}{7}\Bigg|_{\text{at } (3,1,2)}$$

$\therefore$ Directional derivative $= \hat{a}\cdot\nabla\phi$

$$= \frac{2\hat{i}+6\hat{j}+3\hat{k}}{7}\left(-\frac{3\hat{i}+\hat{j}+2\hat{k}}{14\sqrt{14}}\right)$$

$$= -\frac{6+6+6}{7\cdot 14\sqrt{14}} = -\frac{9}{49\sqrt{14}}.$$

Q.3:- Find the directional derivative of $\left(\frac{1}{r^2}\right)$ in the direction of $\vec{r}$, where $\vec{r} = \hat{i}x+\hat{j}y+\hat{k}z$. (2016-17)

Solⁿ:- $\nabla\left(\frac{1}{r^2}\right) = \frac{-2}{r^3}\hat{r} = -\frac{2}{r^4}\vec{r}$ $\left(\because \hat{r} = \frac{\vec{r}}{|\vec{r}|}\right)$

Let $\hat{e}$ be the unit vector in the direction of $\vec{r}$ then $\hat{e} = \frac{\vec{r}}{r}$

Directional derivative $= \nabla\left(\frac{1}{r^2}\right)\cdot\hat{e} = -\frac{2}{r^4}\cdot\vec{r}\cdot\frac{\vec{r}}{r} = -\frac{2}{r^3}$

Q.4:- Find the directional derivative of $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point P(1,1,1) in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$. (2018-19)

Solⁿ:- $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$

$$\therefore \text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(5x^2y - 5y^2z + \frac{5}{2}z^2x\right)$$

$$= \left(10xy + \frac{5}{2}z^2\right)\hat{i} + (5x^2 - 10yz)\hat{j} + (-5y^2 + 5zx)\hat{k}$$

$$= \frac{25}{2}\hat{i} - 5\hat{j} \quad \text{at } (1,1,1)$$

Here $\hat{a} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$

$\therefore$ Directional derivative $= (\text{grad}\,\phi)\cdot\hat{a}$

$$= \left(\frac{25}{2}\hat{i} - 5\hat{j}\right)\cdot\left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right)$$

$$= \frac{25}{3} + \frac{10}{3} = \frac{35}{3}.$$

Q.5:- Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at (1,-2,1) in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$. Find also the greatest rate of increase of ϕ. (2019-20

Solⁿ:- $\phi(x,y,z) = x^2yz + 4xz^2$

$$\therefore \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$$

$$= \hat{i}(2xyz + 4z^2) + \hat{j}(x^2z) + \hat{k}(x^2y + 8xz)$$

At (1,-2,1), $\nabla\phi = \hat{j} + 6\hat{k}$

If $\hat{n}$ is a unit vector in the direction of $2\hat{i}-\hat{j}-2\hat{k}$, then

$$\hat{n} = \frac{2\hat{i}-\hat{j}-2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i}-\hat{j}-2\hat{k})$$

So the required directional derivative at $(1,-2,1)$

$$= \nabla\phi\cdot\hat{n} = (\hat{j}+6\hat{k})\cdot\frac{1}{3}(2\hat{i}-\hat{j}-2\hat{k}) = \frac{-13}{3}$$

Greatest rate of increase of $\phi = |\hat{j}+6\hat{k}| = \sqrt{1+36} = \sqrt{37}$

<u>Notes:-</u> The directional derivative of ϕ is maximum along the normal to the surface i.e., along grad ϕ.

· The maximum value of this directional derivative $= |\nabla f|$

Divergence of a Vector Point Function

Th divergence of a vector point function $\vec{F}$ is denoted
d $\vec{F}$ and is defined as below.

Let $\vec{F} = F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(\hat{i}F_1+\hat{j}F_2+\hat{k}F_3)$$

$$\boxed{\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}}$$

div $\vec{F}$ is a scalar fun

Notes:- (1) If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ then prove that div $\vec{r}$ =

$$\text{div}\,\vec{r} = \nabla\cdot\vec{r} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z) = 1+1+1 = 3.$$

(2) div $\vec{v}$ gives the rate of outflow per unit volume at f the fluid.

(3) A solenoidal vector field is a vector field $\vec{v}$ with d ro at all points in the field i.e. $\boxed{\text{div}\,\vec{v} = \nabla\cdot\vec{v} = 0}$

<u>Q.1</u> :- Show that the vector; $\vec{V} = 3y^4z^2\hat{i}+4x^3z^2\hat{j}-3x^2y^2\hat{k}$ i solenoidal. (201

<u>Sol</u> :- For the vector field $\vec{V}$ to be solenoidal, div $\vec{V}$ =

$$\text{div}\,\vec{V} = \nabla\cdot\vec{V} = \frac{\partial}{\partial x}(3y^4z^2)+\frac{\partial}{\partial y}(4x^3z^2)+\frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0$$

Q.2:- Find the value of m if $\vec{F} = mx\hat{i} - 5y\hat{j} + 2z\hat{k}$ is a solenoidal vector. (2017-18)

Sol:- The vector field $\vec{F}$ to be solenoidal if $\text{div}\,\vec{F}=0$

$\Rightarrow \nabla\cdot\vec{F}=0$

$\Rightarrow \frac{\partial}{\partial x}(mx) + \frac{\partial}{\partial y}(-5y) + \frac{\partial}{\partial z}(2z)=0 \Rightarrow m-5+2=0 \Rightarrow m=3.$

Q.3:- Show that vector $\vec{V} = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is solenoidal. (2019-20, 2020-21)

Sol:- Proceed as Q.1..

Q.4:- Find the divergence of the vector $\vec{F}(x,y,z) = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$.

Sol:-

$$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(xz^3) + \frac{\partial}{\partial y}(-2x^2yz) + \frac{\partial}{\partial z}(2yz^4)$$

$$= z^3 - 2x^2z + 8yz^3$$

Curl of a Vector Point Function

The curl of a vector point function is a vector quantity if $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$. Then the curl (or rotation) of $\vec{V}$ is denoted by curl $\vec{V}$ and is defined as

$$\text{Curl}\,\vec{V} = \nabla\times\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$$

$$\boxed{\text{Curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}}$$

Notes:- (1) The angular velocity at any points is equal to half the curl of linear velocity at that point of the body.

(2) If $\boxed{\text{Curl}\,\vec{V}=0}$, then $\vec{V}$ is said to be an Irrotational vector, otherwise rotational. Also ~~curl of~~ a vector signifies rotation.

Velocity potential:- If a velocity vector $\vec{V}$ is Irrotational then a function ϕ whose gradient is equal to the velocity vector $\vec{V}$, called velocity potential. i.e. $\vec{V} = \nabla\phi$

Vector identities:- (1) $\text{div}(A+B) = \text{div}A + \text{div}B$

(2) $\text{Curl}(A+B) = \text{Curl}A + \text{Curl}B$

(3) If A is differentiable vector function and ϕ is a differentiable scalar function, then $\text{div}(\phi A) = (\text{grad}\,\phi)\cdot A + \phi\,\text{div}A$

(4) $\text{Curl}(\phi A) = (\text{grad}\,\phi)\times A + \phi\,\text{Curl}A$

(5) $\text{div}(A\times B) = B\cdot\text{Curl}A - A\cdot\text{Curl}B$

(6) $\nabla\times(A\times B) = (B\cdot\nabla)A - B(\nabla\cdot A) - (A\cdot\nabla)B + A(\nabla\cdot B)$

(7) $\nabla\times(\nabla f) = 0$

(8) $\nabla\cdot(\nabla\times A) = 0$

(9) $\nabla^2 f \equiv \nabla\cdot(\nabla f)$

Q.1 Imp If $\vec{F} = (\vec{a}\cdot\vec{r})\vec{r}$, where $\vec{a}$ is a constant vector, find curl $\vec{F}$ and prove that it is perpendicular to $\vec{a}$. (2011-12)

Sol^n:→ Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Now $\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$

$$\Rightarrow (\vec{a}\cdot\vec{r})\vec{r} = (a_1x + a_2y + a_3z)\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= (a_1x^2 + a_2xy + a_3xz)\hat{i} + (a_1xy + a_2y^2 + a_3yz)\hat{j} + (a_1xz + a_2yz + a_3z^2)\hat{k}$$

Now
$$\text{Curl}\,(\vec{a}\cdot\vec{r})\cdot\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1x^2+a_2xy+a_3xz & a_1xy+a_2y^2+a_3yz & a_1xz+a_2yz+a_3z^2 \end{vmatrix}$$

$$\text{Curl}\,\vec{F} = \hat{i}(a_2z - a_3y) - \hat{j}(a_1z - a_3x) + \hat{k}(a_1y - a_2x)$$

Now to show curl $\vec{F}$ is perpendicular to $\vec{a}$ i.e. we have to show Curl $\vec{F}\cdot\vec{a} = 0$

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\text{Curl}\,\vec{F}\cdot\vec{a} = (a_2z - a_3y)a_1 - (a_1z - a_3x)a_2 + (a_1y - a_2x)a_3$$

Imp $= 0$

Q.2:→ If $\vec{F} = \frac{\vec{r}}{r^3}$, find curl $\vec{F}$. (2012-13)

Sol^n:→ We know that $\text{curl}(u\vec{a}) = u\,\text{curl}\,\vec{a} + (\text{grad}\,u)\times\vec{a}$

$$\therefore \text{curl}\left(\frac{1}{r^3}\vec{r}\right) = \frac{1}{r^3}\,\text{curl}\,\vec{r} + \left(\text{grad}\,\frac{1}{r^3}\right)\times\vec{r}$$

$$= \frac{1}{r^3}(\vec{0}) + \left(-\frac{3}{r^4}\vec{r}\right)\times\vec{r} \qquad [\because \text{curl}\,\vec{r} = 0]$$

$$= \vec{0} - \frac{3}{r^5}(\vec{r}\times\vec{r}) = \vec{0} - \vec{0} = \vec{0}.$$

Q.:→ Prove that $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)$ is irrotational. (20

Sol^n:→ Field $\vec{A}$ is irrotational if curl $\vec{A} = \vec{0}$

Now
$$\text{Curl}\,\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$= \hat{i}(-1+1) - \hat{j}(3z^2 - 3z^2) + \hat{k}(6x - 6x) =$$

Q. :→ Find the curl of $\vec{F} = xy\hat{i} + y^2\hat{j} + xz\hat{k}$ at $(-2,4,1)$. (20

Sol :→
$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 & xz \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(z-0) + \hat{k}(0-x) = -z\hat{j} -$$

At $(-2,4,1)$, Curl $\vec{F} = -\hat{j} + 2\hat{k}$

Q. :→ Prove that, for every field $\vec{V}$, div curl $\vec{V} = 0$.

Sol :→ Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

then
$$\text{curl}\,\vec{V} = \nabla\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right\} - \hat{j}\left\{\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right\} + \hat{k}\left\{\frac{\partial V_2}{\partial x}\right.$$

$$\text{div.}(\text{Curl}\,\vec{V}) = \nabla\cdot(\nabla\times\vec{V})$$

$$= \frac{\partial}{\partial x}\left\{\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right\} + \frac{\partial}{\partial z}\left\{\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right\}$$

$$\nabla \cdot \nabla \times \vec{v} = \left(\frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 V_1}{\partial y \partial z} - \frac{\partial^2 V_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y}\right) = 0$$

Q.6:→ A fluid motion is given by $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$. Show that the motion is irrotational and hence find the velocity potential. (2015 16)

Solⁿ:→ $\text{Curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = (1-1)\hat{i} - (1-1)\hat{j} + (1-1)\hat{k} = 0$

Hence $\vec{V}$ is irrotational.

To find the corresponding velocity potential ϕ, consider the following relation.

$$\vec{V} = \nabla\phi$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi \cdot d\vec{r} = \nabla\phi \cdot d\vec{r} = \vec{V} \cdot d\vec{r}$$

$$= [(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}] \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= (y+z)dx + (z+x)dy + (x+y)dz$$

$$= y\,dx + z\,dx + z\,dy + x\,dy + x\,dz + y\,dz$$

$$\Rightarrow \phi = \int(y\,dx + x\,dy) + \int(z\,dy + y\,dz) + \int(z\,dx + x\,dz)$$

$$\Rightarrow \phi = xy + yz + zx + c.$$

velocity potential $= xy + yz + zx + c$

Q.7:→ If $\vec{A} = (xz^2\hat{i} + 2y\hat{j} - 3xz\hat{k})$ and $\vec{B} = (3xz\hat{i} + 2yz\hat{j} - z^2\hat{k})$. Find the value of $[\vec{A} \times (\nabla \times \vec{B})]$ & $[(\vec{A} \times \nabla) \times \vec{B}]$. (2016-17)

Solⁿ:→ (i) $[\vec{A} \times (\nabla \times \vec{B})]$

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz & 2yz & -z^2 \end{vmatrix} = \hat{i}[0-2y] - \hat{j}[0-3x] + \hat{k}[0-0]$$

$$= -2y\hat{i} + 3x\hat{j} + 0\hat{k}$$

Now

$$[\vec{A} \times (\nabla \times \vec{B})] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xz^2 & 2y & -3xz \\ -2y & 3x & 0 \end{vmatrix} = \hat{i}[0 + 9x^2z] - \hat{j}[0 - 6xyz] + \hat{k}[3x^2z^2 + 4y^2]$$

$$= 9x^2z\,\hat{i} + 6xyz\,\hat{j} + (3x^2z^2 + 4y^2)\hat{k}$$

(ii) $[(\vec{A} \times \nabla) \times \vec{B}]$

$$\vec{A} \times \nabla = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xz^2 & 2y & -3xz \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \hat{i}[0+0] - \hat{j}[2xz + 3z] + \hat{k}[0-0]$$

$$= 0\hat{i} - (2xz + 3z)\hat{j} + 0\hat{k}$$

$$[(\vec{A} \times \nabla) \times \vec{B}] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -(2xz+3z) & 0 \\ 3xz & 2yz & -z^2 \end{vmatrix} = [z^2(2xz + 3z) - 0]\hat{i} - [0-0]\hat{j} + [0 + 3xz(2xz + 3z)]\hat{k}$$

$$= (2xz^3 + 3z^3)\hat{i} + (6x^2z^2 + 9xz^2)\hat{k}$$

Q.8:→ Determine the value of constants a, b, c if $\vec{F} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$ is irrotational. (2017-18)

Solⁿ:→ It is given that $\vec{F}$ is irrotational i.e. $\text{Curl}\vec{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\Rightarrow (c+1)\hat{i} - (4-a)\hat{j} + (b-2)\hat{k} = 0.$$

Now $c+1=0 \Rightarrow c=-1$

$4-a=0 \Rightarrow a=4$

$b-2=0 \Rightarrow b=2$

Q.9:- If all second order derivatives of ϕ and $\vec{v}$ are continuous, then show that (i) $Curl(grad\phi)=0$ (ii) $div(curl\vec{v})=0$ (2017-18)

Sol^n:- (i) $grad\,\phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$$curl(grad\,\phi) = \nabla\times(\nabla\phi) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \hat{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + \hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$$

$$= \vec{0}.$$

(ii) Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$curl\,\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \hat{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$div(curl\,\vec{v}) = \nabla\cdot(\nabla\times\vec{v})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[\hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \hat{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\right]$$

$$= \frac{\partial^2 v_3}{\partial x\partial y} - \frac{\partial^2 v_2}{\partial x\partial z} + \frac{\partial^2 v_1}{\partial y\partial z} - \frac{\partial^2 v_3}{\partial y\partial x} + \frac{\partial^2 v_2}{\partial z\partial x} -$$

$$= 0.$$

Q.1:- Prove that $(y^2z^2+3yz-2x)\hat{i} + (3xz+2xy)\hat{j}+($

is both solenoidal and irrotational.

Sol^n:- Proceed as Q.1 in $\overset{divergence}{\downarrow}$ for solenoidal and proce

Q.3 for irrotational. in curl section.

Q. :- A fluid motion is given by

$\vec{v} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)$

Is the motion irrotational? If so, find the velocity p

Sol :- Proceed as

Q. :- If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, prove that $Curl\,\vec{r} = \vec{0}$ i.e., $\nabla\times\vec{r}$

Sol :- $Curl\,\vec{r} = \nabla\times\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(x\hat{i}+y\hat{j}+$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{j}\left(\frac{\partial x}{\partial z}\right. + \hat{k}\left(\frac{\partial y}{\partial x}\right.$$

$$= 0$$

VECTOR INTEGRAL CALCULUS

ne Integral:- Let $\vec{F}(x,y,z)$ be a vector function and a curve AB, Line integral of a vector function $\vec{F}$ along the curve AB is defined as

$$\boxed{\text{Line integral} = \int_C \left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right) ds = \int_C \vec{F}\cdot d\vec{r}}$$

ork:- If $\vec{F}$ represents the variable force acting on a particle along arc AB, then the $\boxed{\text{total work done} = \int_A^B \vec{F}\cdot d\vec{r}}$.

1:- Find the work done in moving a particle in the force field: $\vec{F} = 3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}$ along the curve $x^2=4y$ and $3x^3 = 8z$ from $x=0$ to $x=2$. (2011-)

Solⁿ:- Work done $= \int_C \vec{F}\cdot d\vec{r}$

$$= \int_C [3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}]\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C 3x^2 dx + (2xz-y)dy + z\,dz$$

along the curve $x^2=4y$ and $3x^3=8z$ from $x=0$ to $x=2$

$\Rightarrow$ $2x\,dx = 4dy$ and $9x^2dx = 8\,dz$

$$\text{Work done} = \int_{x=0}^{2} \left[3x^2dx + \left(2x\cdot\frac{3x^3}{8} - \frac{x^2}{4}\right)\frac{x}{2}dx + \frac{3x^3}{8}\cdot\frac{9x}{8}dx\right]$$

$$= \int_{x=0}^{2}\left[3x^2 + \frac{1}{8}(3x^5 - x^3) + \frac{27}{64}x^5\right]dx$$

$$= \left[x^3 + \frac{1}{8}\left(\frac{3x^6}{6} - \frac{x^4}{4}\right) + \frac{27}{64}\frac{x^6}{6}\right]_0^2$$

$$= \left[8 + \frac{1}{8}(32-4) + \frac{27}{64}\cdot\frac{64}{6}\right] = 16$$

Imp

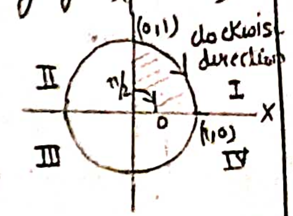
Ques:- Evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the curve $x^2+y^2=1$, $z=1$ in the positive direction from $(0,1,1)$ to $(1,0,1)$, where: $\vec{F} = (yz+2x)\hat{i} + xz\hat{j} + (xy+2z)\hat{k}$. (2012-13)

Solⁿ:- $\int_C \vec{F}\cdot d\vec{r} = \int_C [(yz+2x)dx + xz\,dy + (xy+2z)dz]$

$\because$ $z=1$ hence $dz=0$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_C (y+2x)dx + x\,dy = \int_C [(x\,dy + y\,dx) + 2x\,dx]$$

$$= \int_C d(xy) + \int_C 2x\,dx$$



The parametric equations of given path $x^2+y^2=1$ are $x=\cos\theta$ and $y=\sin\theta$

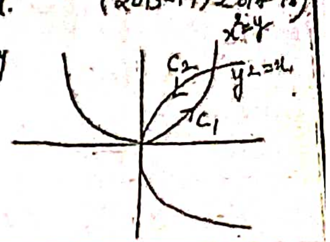
$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{\pi/2}^{0} d[\cos\theta\sin\theta] + \int_{\pi/2}^{0} 2\cdot\cos\theta\cdot(-\sin\theta)\,d\theta$$

$$= [\cos\theta\sin\theta]_{\pi/2}^{0} + \int_{\pi/2}^{0}\sin2\theta\,d\theta$$

$$= 0 - \left[-\frac{\cos2\theta}{2}\right]_{\pi/2}^{0} = \frac{1}{2}[\cos0 - \cos\pi] = 1$$

Q.31:- If $\vec{A} = (x-y)\hat{i} + (x+y)\hat{j}$, evaluate $\oint_C \vec{A}\cdot d\vec{r}$ around the curve C consisting of $y=x^2$ and $y^2=x$. (2013-14, 2017-18)

Solⁿ:- $\oint_C \vec{A}\cdot d\vec{r} = \oint_C (x-y)dx + (x+y)dy$



C consisting of $y=x^2$ and $y^2=x$

$$\oint_C \vec{A}\cdot d\vec{r} = \oint_{C_1}\vec{A}\cdot d\vec{r} + \oint_{C_2}\vec{A}\cdot d\vec{r}$$

Along C_1, $y=x^2$, $dy=2x\,dx$

⇒ $\oint_{C_1}\vec{A}\cdot d\vec{r}=\int_0^1(x-x^2)\,dx+(x+x^2)\,2x\,dx=\int_0^1(x+x^2+2x^3)\,dx$

$$=\left[\frac{x^2}{2}+\frac{x^3}{3}+\frac{2x^4}{4}\right]_0^1=\frac{1}{2}+\frac{1}{3}+\frac{1}{2}=\frac{4}{3}$$

Now Along C_2, $x=y^2$, $dx=2y\,dy$

$$\oint_{C_2}\vec{A}\cdot d\vec{r}=\int_1^0(y^2-y)\,2y\,dy+(y^2+y)\,dy$$

$$\oint_{C_2}\vec{A}\cdot d\vec{r}=\int_1^0(2y^3-y^2+y)\,dy=\left[\frac{y^4}{2}-\frac{y^3}{3}+\frac{y^2}{2}\right]_1^0$$

$$=-\left[\frac{1}{2}-\frac{1}{3}+\frac{1}{2}\right]=-\frac{2}{3}$$

Thus $\oint_{C}\vec{A}\cdot d\vec{r}=\frac{4}{3}+\left(-\frac{2}{3}\right)=\frac{2}{3}$

Surface Integral

Surface integral of a vector function $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface.

Component of $\vec{F}$ along the normal $=\vec{F}\cdot\hat{n}$, where $\hat{n}$ is the unit normal vector to an element ds and

$\hat{n}=\dfrac{\text{grad} f}{|\text{grad} f|}$ $\quad ds=\dfrac{dx\,dy}{|(\hat{n}\cdot\hat{k})|}$ (normal to the surface (x-y) plane)

Surface integral of f over S $=\Sigma\vec{F}\cdot\hat{n}=\iint_S(\vec{F}\cdot\hat{n})\,ds$

where

$ds=\dfrac{dy\,dz}{|(\hat{n}\cdot\hat{i})|}$ (normal to the surface y-z plane)

$ds=\dfrac{dz\,dx}{|\hat{n}\cdot\hat{j}|}$ (normal to the surface z-x plane)

NP'

Q 4(b) Evaluate: $\iint_S\vec{F}\cdot\hat{n}\,ds$ where $\vec{F}=18z\hat{i}-12\hat{j}+3y\hat{k}$ an
the part of the plane $2x+3y+6z=12$ in the first

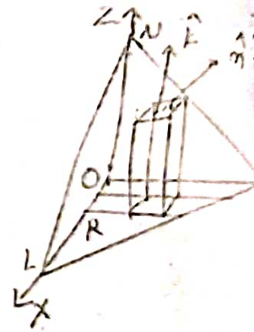
Sol:- A vector normal to the surface S is given by

$$\nabla(2x+3y+6z)=2\hat{i}+3\hat{j}+6\hat{k}$$

∴ $\hat{n}$ = a unit vector normal to surface S $=\dfrac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{(2)^2+(3)^2+}}$

$$=\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}$$

$$\hat{k}\cdot\hat{n}=\hat{k}\cdot\left(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}\right)=\frac{6}{7}$$



∴ $\iint_S\vec{A}\cdot\hat{n}\,ds=\iint_R\vec{A}\cdot\hat{n}\,\dfrac{dx\,dy}{|\hat{k}\cdot\hat{n}|}$

where R is the projection of S i.e. triangle LMN on the xy-plane. The region R, i.e., triangle OLM is bounded by x-axis, y-axis and the line $2x+3y=12$, $z=0$.

Now $\vec{A}\cdot\hat{n}=\left[18z\hat{i}-12\hat{j}+3y\hat{k}\right]\cdot\left(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}\right)$

$$=\frac{36}{7}z-\frac{36}{7}+\frac{18}{7}y$$

$$=\frac{36}{7}\left(\frac{12-2x-3y}{6}\right)-\frac{36}{7}+\frac{18}{7}y \qquad \left[\because \text{ on the plane } 2x+ \quad z=\frac{12-2x-3y}{6}\right.$$

$$=\frac{-12x}{7}+\frac{36}{7}$$

Hence $\iint_S\vec{F}\cdot\hat{n}\,ds=\iint_R\vec{F}\cdot\hat{n}\,\dfrac{dx\,dy}{|\hat{k}\cdot\hat{n}|}=\iint_R\left(\dfrac{-12x}{7}+\dfrac{36}{7}\right)\dfrac{7}{6}\,dx\,dy$

$$=\iint_R(-2x+6)\,dy\,dx$$

$$= -2\int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}}(x-3)\;dy\,dx = 2\int_{x=0}^{6}(3-x)[y]_0^{\frac{12-2x}{3}}dx$$

$$= 2\int_{x=0}^{6}(3-x)\left(\frac{12-2x}{3}\right)dx = \frac{2}{3}\int_0^6(2x^2-18x+36)\,dx$$

$$= \frac{2}{3}\left[\frac{2}{3}x^3-9x^2+36x\right]_0^6$$

$$= \;=\; = 24$$

Q.56→ Evaluate $\iint_S (yz\hat{i}+2x\hat{j}+xy\hat{k})\cdot ds$, where S is the surface of the sphere $x^2+y^2+z^2=a^2$ in the first octant. (2014-1.)

Solⁿ:→ $\phi = x^2+y^2+z^2-a^2$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

$$\vec{A} = yz\hat{i}+zx\hat{j}+xy\hat{k}$$

$$\Rightarrow \vec{A}\cdot\hat{n} = \frac{xyz+xyz+xyz}{a} = \frac{3xyz}{a}$$

$$\hat{n}\cdot\hat{k} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\cdot\hat{k} = \frac{z}{a}$$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \iint\frac{3xyz}{a}\,\frac{dx\,dy}{z/a} = 3\iint xy\,dx\,dy$$

$$= 3\int_0^a\int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx = \frac{3}{2}\int_0^a x\,[y^2]_0^{\sqrt{a^2-x^2}}dx$$

$$= \frac{3}{2}\int_0^a x(\sqrt{a^2-x^2})^2dx = \frac{3}{2}\int_0^a x(a^2-x^2)\,dx$$

$$= \frac{3}{2}\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_0^a = \frac{3}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right] = \frac{3}{8}a^4$$

Volume Integral

Let $\vec{F}$ be a vector point function and volume V enclosed by a closed surface. The $\boxed{\text{Volume Integral} = \iiint_V \vec{F}\,dV}$

Ques→ If $\vec{F} = 2z\hat{i}-x\hat{j}+y\hat{k}$, evaluate $\iiint_V \vec{F}\,dv$, where v is the region bounded by the surfaces $x=0, y=0, x=2, y=4, z=x^2, z=2$.

Solⁿ:→ $\iiint_V \vec{F}\,dv = \iiint(2z\hat{i}-x\hat{j}+y\hat{k})\,dx\,dy\,dz$

$$= \int_0^2\int_0^4\int_{x^2}^{2}(2z\hat{i}-x\hat{j}+y\hat{k})\,dz\,dy\,dx$$

$$= \int_0^2\int_0^4\left[z^2\hat{i}-xz\hat{j}+yz\hat{k}\right]_{x^2}^{2}dy\,dx$$

$$= \int_0^2\int_0^4\left[4\hat{i}-2x\hat{j}+2y\hat{k}-x^4\hat{i}+x^3\hat{j}-x^2y\hat{k}\right]dy\,dx$$

$$= \int_0^2\left[4y\hat{i}-2xy\hat{j}+y^2\hat{k}-x^4y\hat{i}+x^3y\hat{j}-\frac{x^2y^2}{2}\hat{k}\right]_0^4dx$$

$$= \int_0^2(16\hat{i}-8x\hat{j}+16\hat{k}-4x^4\hat{i}+4x^3\hat{j}-8x^2\hat{k})\,dx$$

$$= \left[16x\hat{i}-4x^2\hat{j}+16x\hat{k}-\frac{4x^5}{5}\hat{i}+x^4\hat{j}-\frac{8x^3}{3}\hat{k}\right]_0^2$$

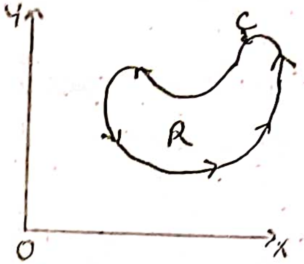
$$= 32\hat{i}-16\hat{j}+32\hat{k}-\frac{128}{5}\hat{i}+16\hat{j}-\frac{64}{3}\hat{k} = \frac{32\hat{i}}{5}+\frac{32\hat{k}}{3} = \frac{32}{15}(3\hat{i}+$$

Green's Theorem

(2011-12, 2020-21)

If $\phi(x,y)$, $\psi(x,y)$, $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \psi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in x-y plane, then

$$\boxed{\oint_C (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy}$$



Q.1:- Using Green's theorem, evaluate the integral $\oint_C (xy\,dy - y^2 dx)$, where C is the square cut-from the first quadrant by the lines $x=1$, $y=1$.

(2012-13)

Sol:- By Green's theorem

$$\oint_C xy\,dy - y^2 dx = \iint_R \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(xy)\right] dx\,dy$$

$$= \iint_R (0 - x)\,dx\,dy = -\int_{x=0}^{1}\int_{y=0}^{1} x\,dx\,dy$$

$$= -\int_{x=0}^{1} x\,[y]_0^1\,dx = -\left[\frac{x^2}{2}\right]_0^1 = -\frac{1}{2}$$

Imp.

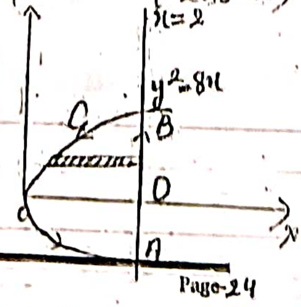
Q.2:- Verify Green's theorem in plane for: $\oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy$, where C is the boundary of the region defined by $y^2=8x$ and $x=2$.

(2013-14)

Sol:- Using Green's theorem

$$\oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2y+3) - \frac{\partial}{\partial y}(x^2-2xy)\right] dx\,dy$$



$$= \iint_R (2xy + 2x)\,dx\,dy = \int_{y=-4}^{4}\int_{x=y^2/8}^{2} (2xy+2x)\,dx\,dy$$

$$\int_{y=-4}^{4} \left[x^2y + x^2\right]_{x=y^2/8}^{2} dy = \int_{y=-4}^{4} \left[4y + 4 - \frac{y^5}{64} - \frac{y^4}{64}\right.$$

$$= \left[2y^2 + 4y - \frac{y^6}{64\times6} - \frac{y^5}{64\times5}\right]_{-4}^{4} = \frac{128}{5}$$

Verification of the Green's theorem:

$$\oint (x^2-2xy)\,dx + (x^2y+3)\,dy = \left[\int_{BOA} [(x^2-2xy)\,dx + (x^2\right.$$

$$+ \left.\int_{ADB} [(x^2-2xy)\,dx + (x^2y+3)\,dy]\right]$$

Along BOA, $y^2=8x$, $dx = \frac{y}{4}dy$

Along ADB, $x=2$, $dx=0$

$$= \int_{y=4}^{-4} \left[\left(\frac{y^4}{64} - \frac{y^3}{4}\right)\frac{y}{4}\,dy + \left(\frac{y^5}{64}+3\right)dy\right]$$

$$+ \int_{y=-4}^{4} [(4-4y)\cdot 0 + (4y+3)\,dy]$$

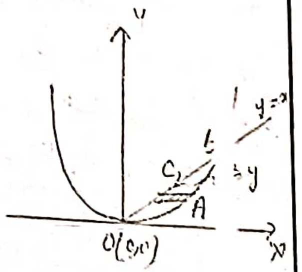
$$= \left[\frac{y^6}{64\times24} - \frac{y^5}{16\times5} + \frac{y^6}{64\times6} + 3y\right]_4^{-4} + \left[2y^2+3y\right]_{-4}^{4}$$

$$= \frac{128}{5} - 24 + 24 = \frac{128}{5}$$

Verified.

:-1 Verify the Green's theorem to evaluate the line integral $(2y^2dx+3xdy)$, where C is the boundary of the closed region ...nded by $y=x$ and $y=x^2$. (2015-)

:-1 Using Green's theorem



$$2y^2dx+3xdy = \iint\left[\frac{\partial}{\partial x}(3x)-\frac{\partial}{\partial y}(2y^2)\right]dxdy$$

$$= \iint(3-4y)dxdy = \int_0^1\int_{x=y}^{\sqrt{y}}(3-4y)\,dxdy$$

$$\int_0^1[3x-4xy]_y^{\sqrt{y}}\,dy = \int_0^1(3\sqrt{y}-4y^{3/2}-3y+4y^2)dy$$

$$= \left[3\times\frac{2}{3}y^{3/2}-4\times\frac{2}{5}y^{5/2}-\frac{3y^2}{2}+\frac{4y^3}{3}\right]_0^1 = \frac{6}{3}-\frac{8}{5}-\frac{3}{2}+\frac{4}{3} = \frac{7}{30}$$

Verification of the Green's theorem:

$$2y^2dx+3xdy = \int_{OAB}(2y^2dx+3xdy)+\int_{BCO}(2y^2dx+3xdy)$$

Along OAB, $y=x^2$, $dy=2xdx$

Along BCO, $y=x$, $dy=dx$

$$= \int_0^1[2x^4+3x(2x)]dx+\int_1^0(2x^2+3x)dx$$

$$= \left[\frac{2x^5}{5}+2x^3\right]_0^1+\left[\frac{2x^3}{3}+\frac{3x^2}{2}\right]_1^0$$

$$= \left(\frac{2}{5}+2\right)-\left(\frac{2}{3}+\frac{3}{2}\right) = \frac{12}{5}-\frac{13}{6} = \frac{72-65}{30} = \frac{7}{30}$$ Verified

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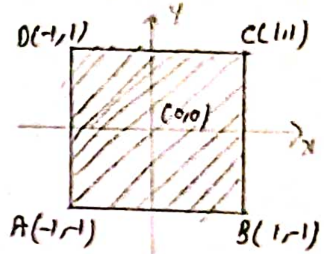
Q.4:-1 Verify Green's theorem, evaluate $\int_C(x^2+xy)dx+(x^2+y^2)dy$, where C square formed by lines $x=\pm1, y=\pm1$. (2017-18)

Sol^n:-1 By Green's theorem, we have



$$\int_C(x^2+xy)dx+(x^2+y^2)dy = \iint_R\left[\frac{\partial}{\partial x}(x^2+y^2)-\frac{\partial}{\partial y}(x^2+xy)\right]dxdy$$

$$= \iint_R(2x-x)\,dxdy = \int_{x=-1}^{1}\int_{y=-1}^{1}x\,dxdy = \int_{-1}^{1}x\,(y)_{-1}^{1}dx = \int_{-1}^{1}2xdx$$

$$= 0$$

Verification of the Green's theorem:

$$\int_C[(x^2+xy)dx+(x^2+y^2)dy] = \int_{AB}[(x^2+xy)dx+(x^2+y^2)dy]+$$

$$\int_{BC}[(x^2+xy)dx+(x^2+y^2)dy]+\int_{CD}[(x^2+xy)dx+(x^2+y^2)dy]$$

$$+\int_{DA}[(x^2+xy)dx+(x^2+y^2)dy]$$

Now along AB, $y=-1$ and $dy=0$.

$$\therefore \int_{AB}[(x^2+xy)dx+(x^2+y^2)dy] = \int_{-1}^{1}(x^2-x)dx = \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{-1}^{1}$$

$$= \left[\frac{1}{3}-\frac{1}{2}+\frac{1}{3}+\frac{1}{2}\right] = \frac{2}{3}$$

Along BC, $x=1$ and $dx=0$

$$\therefore \int_{BC}[(x^2+xy)dx+(x^2+y^2)dy] = \int_{-1}^{1}(1+y^2)dy = \left[y+\frac{y^3}{3}\right]_{-1}^{1} = \frac{8}{3}$$

Along CD, $y=1$ and $dy=0$

$$\therefore \int_{CD}[(x^2+xy)dx+(x^2+y^2)dy] = \int_1^{-1}(x^2+x)\,dx = \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_1^{-1}$$

$$= \left[\frac{-1}{3}+\frac{1}{2}-\frac{1}{3}-\frac{1}{2}\right] = -\frac{2}{3}$$

Along DA, $x=-1$. and $dx=0$

$$\therefore \int_{DA}[(x^2+xy)dx+(x^2+y^2)dy] = \int_1^{-1}(1+y^2)\,dy = \left[y+\frac{y^3}{3}\right]_1^{-1}$$

$$= \left[-1-\frac{1}{3}-1-\frac{1}{3}\right] = -\frac{8}{3}$$

$$\Rightarrow \int_C[(x^2+xy)dx+(x^2+y^2)dy] = \frac{2}{3}+\frac{8}{3}-\frac{2}{3}-\frac{8}{3} = 0$$ Verified

Area of the Plane region by Green's Theorem:→

$$\boxed{\text{Area} = \frac{1}{2}\oint_C (x\,dy - y\,dx)}$$

where C is a closed curve

Stoke's Theorem

Surface Integral of the component of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve C. Mathematically

$$\boxed{\oint \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds}$$ where $ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

where $\hat{n}$ is a unit external normal to the surface.

Note:- Stokes' theorem relates a surface integral over a surface S to a line integral around the boundary of S (a space curve).

Q.1 → Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ by Stoke's theorem, where:

$\bar{F} = y^2\hat{i}+x^2\hat{j}-(x+z)\hat{k}$ and C is the boundary of triangle vertices at (0,0,0), (1,0,0) and (1,1,0).

Sol^n → Since z-coordinates of each vertex of the triangle is zero, therefore, the triangle lies in the xy-plane and $\hat{n}=\hat{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = \hat{j}+2(x-y)\hat{k}$$

$$\therefore \text{curl}\,\vec{F}\cdot\hat{n} = [\hat{j}+2(x-y)\hat{k}]\cdot\hat{k} = 2(x-y)$$

The equation of line OB is $y=x$.

Stoke's theorem,

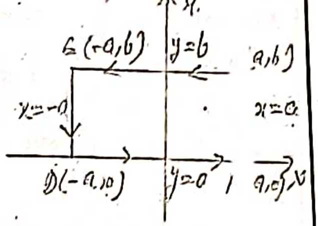
$$\oint_c \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$= \int_0^1\int_0^x 2(x-y)\,dy\,dx = \int_0^1 2\left[xy-\frac{y^2}{2}\right]_0^x dx$$

$$= 2\int_0^1\left(x^2-\frac{x^2}{2}\right)dx = \int_0^1 x^2\,dx = \frac{1}{3}.$$

2:- Verify Stokes theorem for $\vec{F}=(x^2+y^2)\hat{i}-2xy\hat{j}$ taken around rectangle bounded by the lines $x=\pm a$, $y=0$ and $y=b$. (2017-18) (2014-15)

Let C denote the boundary of the rectangle ABED, then



$$\vec{F}\cdot d\vec{r} = \oint_c [(x^2+y^2)\hat{i} - 2xy\hat{j}]\cdot(\hat{i}dx+\hat{j}dy).$$

$$= \oint_c [(x^2+y^2)dx - 2xy\,dy]$$

curve C consists of four lines AB, BE, ED and DA.

ong AB, $x=a$, $dx=0$ and y varies from 0 to b.

$$\int_{AB}[(x^2+y^2)dx-2xy\,dy] = \int_0^b -2ay\,dy = -a[y^2]_0^b = -ab \quad \text{—①}$$

ong BE, $y=b$, $dy=0$ and x varies from a to -a.

$$\int_{BE}[(x^2+y^2)dx-2xy\,dy] = \int_a^{-a}(x^2+b^2)dx = \left[\frac{x^3}{3}+b^2x\right]_a^{-a}$$

$$= \frac{-2a^3}{3} - 2ab^2 \quad \text{—②}$$

long ED, $x=-a$, $dx=0$ and y varies from b to 0.

$$\therefore \int_{ED}[(x^2+y^2)dx-2xy\,dy] = \int_b^0 2ay\,dy = a[y^2]_b^0 = -ab^2 \quad \text{—③}$$

Along DA, $y=0$, $dy=0$ and x varies from -a to a.

$$\therefore \int_{DA}[(x^2+\ )dx-2\ \ ] = \int_{-a}^{a} x^2\,dx = \frac{2a^3}{3} \quad \text{—④}$$

Adding ①, ②, ③ and ④, we get

$$\oint_c \vec{F}\cdot d\vec{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2 \quad \text{—⑤}$$

Now

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = (-2y-2y)\hat{k} = -4y\hat{k}$$

For the surface S, $\hat{n}=\hat{k}$

$$\text{curl}\,\vec{F}\cdot\hat{n} = -4y\hat{k}\cdot\hat{k} = -4y$$

$$\therefore \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int_0^b\int_{-a}^{a} -4y\,dx\,dy = \int_0^b -4y\,[x]_{-a}^{a}\,dy$$

$$= -8a\int_0^b y\,dy = -8a\left[\frac{y^2}{2}\right]_0^b = -4ab^2 \quad \text{—⑥}$$

The equality of ⑤ and ⑥ verifies Stoke's theorem.

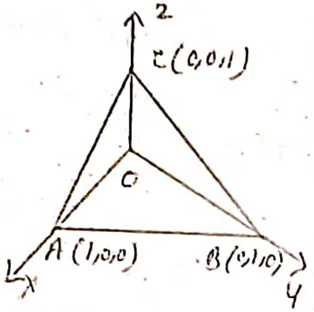
Q.3:- Verify Stokes theorem $\vec{F}=(2y+z, x-z, y-x)$ taken over the triangle ABC cut from the plane $x+y+z=1$ by the coordinate planes. (2016-17)

Sol:- By Stoke's theorem

$$\oint_c \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds \quad \text{—①}$$

Taking LHS,

$$\oint_{ABC} \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CA} \vec{F}\cdot d\vec{r}$$



Along AB, $z=0, x+y=1, y=1-x, dy=-dx$
and $\vec{r} = x\hat{i}+y\hat{j}$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{AB} [2y\hat{i}+x\hat{j}+(y-x)\hat{k}](\hat{i}dx+\hat{j}dy)$$

$$= \int_{AB} 2y\,dx + x\,dy = \int_{AB} 2(1-x)dx + x(-dx)$$

$$[\because y=1-x \text{ and } dy=-dx]$$

$$= \int_{AB} 2\,dx - 2x\,dx - x\,dx = \int_1^0 (2-3x)dx = \left[2x-\frac{3x^2}{2}\right]_1^0 = \frac{-1}{2}$$

Along BC, $x=0, y+z=1, z=1-y, dz=-dy$ and $\vec{r} = y\hat{j}+z\hat{k}$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_{BC} [(2y+z)\hat{i}-z\hat{j}+y\hat{k}](\hat{j}dy+\hat{k}dz)$$

$$= \int_{BC} -z\,dy + y\,dz = \int_{BC} -(1-y)dy + y(-dy) = \int_{BC} -dy + y\,dy - y\,dy$$

$$= \int_1^0 -dy = [-y]_1^0 = 0-(-1) = 1.$$

Along CA, $y=0, x+z=1 \Rightarrow x=1-z, dx=-dz$ and $\vec{r} = x\hat{i}+z\hat{k}$

$$\int_{CA} \vec{F}\cdot d\vec{r} = \int_{CA} [z\hat{i}+(x-z)\hat{j}-x\hat{k}](\hat{i}dx+\hat{k}dz) = \int_{CA} z\,dx - x\,dz$$

$$= \int_{CA} z(-dz)-(1-z)dz = \int_{AC} -z\,dz - dz + z\,dz = \int_{AC} -dz$$

$$= \int_1^0 -dz = -[z]_1^0 = -[0-1] = 1$$

Hence $\int_{AB} \vec{F}\cdot d\vec{r} = \frac{-1}{2} + 1 + 1 = \frac{3}{2}$

Now, curl $\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y+z & x-z & y-x \end{vmatrix}$

$$= \hat{i}[1-(-1)] - \hat{j}[-1-1] + \hat{k}[1-2] = 2\hat{i}+2\hat{j}-\hat{k}$$

Equation of the plane ABC is $x+y+z=1$

Normal to the plane ABC is

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x+y+z-1) = \hat{i}+\hat{j}+\hat{k}$$

Normal unit vector, $\hat{n} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

Now, taking RHS of eqn ①

$$\iint \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_S (2\hat{i}+2\hat{j}-\hat{k})\left(\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}} \quad \frac{dx\,dy}{\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})\hat{k}}\right)$$

$$= \iint_S \frac{2+2-1}{\sqrt{3}} \frac{dx\,dy}{1/\sqrt{3}} = 3\iint dx\,dy = 3\times \text{area of } \Delta$$

$$= 3\times\frac{1}{2}\times1\times1 = \frac{3}{2}$$

Verified

Q.17 Verify Stoke's theorem for the vector field $\vec{F} = (x^2-y^2)\hat{i}$ integrated round the rectangle in the plane $z=0$ and bounded by the lines $x=0, y=0, x=a, y=b$.

(2019-

Sol: Proceed as Q.2.

S8:- Verify Stoke's theorem for the function $\vec{F} = x^2\hat{i} + xy\hat{j}$ ntegrated round the square whose sides are $x=0, y=0,$ $-a, y=a$ in the plane $z=0$. (2020-21)

n:- Proceed as Q.2.

Gauss Divergence Theorem

The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S. Mathematically

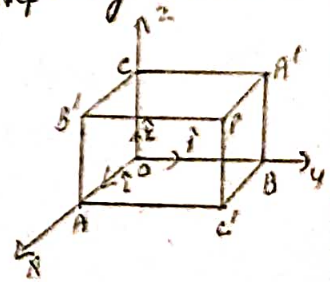
$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv}$$

where $\hat{n}$ is the outward drawn unit normal vector to the surface S.

Imp

Q.1:- Verify the Gauss divergence theorem for: $\vec{F} = (x^2-yz)\hat{i} + (y^2-xz)\hat{j} + (z^2-xy)\hat{k}$ taken over the rectangular parallelepiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$. (2012-13)

Sol^n:- For verification of divergence therem, we shall evaluate the volume and surface integrals separately and show that they are equal.



Now $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-2x)$
$+\frac{\partial}{\partial z}(z^2-xy)$
$= 2(x+y+z)$

$$\therefore \iiint_V \operatorname{div}\vec{F}\,dV = \int_0^c\int_0^b\int_0^a 2(x+y+z)\,dx\,dy\,dz$$

$$= \int_0^c\int_0^b 2\left[\left(\frac{x^2}{2}+yx+zx\right)\right]_0^a dy\,dz$$

$$= \int_0^c\int_0^b 2\left(\frac{a^2}{2}+ya+za\right)dy\,dz = \int_0^c 2\left[\left(\frac{a^2}{2}y+\frac{ay^2}{2}+azy\right)\right]_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right)dz = 2\left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right]_0^c$$

$$= a^2bc + ab^2c + abc^2 = abc(a+b+c)$$

To evaluate the surface integral, divide the closed surface S of the rectangular parallelopiped into 6 parts.

S_1: the face $OAC'B$, S_2: the face $CB'PA'$, S_3: the face $OBA'C$, S_4: the face $AC'PB'$, S_5: the face $OCB'A$, S_6: the face $BA'PC'$.

Also $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \vec{F}\cdot\hat{n}\,ds + \iint_{S_2} \vec{F}\cdot\hat{n}\,ds + \iint_{S_3} \vec{F}\cdot\hat{n}\,ds$

$$+ \iint_{S_4} \vec{F}\cdot\hat{n}\,ds + \iint_{S_5} \vec{F}\cdot\hat{n}\,ds + \iint_{S_6} \vec{F}\cdot\hat{n}\,ds$$

On S_1 $(z=0)$, we have $\hat{n} = -\hat{k}$, $\vec{F} = x^2\hat{i} + y^2\hat{j} - xy\hat{k}$

So that $\vec{F}\cdot\hat{n} = (x^2\hat{i} + y^2\hat{j} - xy\hat{k})\cdot(-\hat{k}) = xy$

$$\therefore \iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a xy\,dx\,dy = \int_0^b \left[y\frac{x^2}{2}\right]_0^a dy = \frac{a^2}{2}\int_0^b y\,dy$$

$$= \frac{a^2b^2}{4}$$

On S_2 $(z=c)$, we have $\hat{n} = \hat{k}$, $\vec{F} = (x^2-cy)\hat{i} + (y^2-cx)\hat{j} + (c^2-xy)\hat{k}$

So that $\vec{F}\cdot\hat{n} = [(x^2-cy)\hat{i} + (y^2-cx)\hat{j} + (c^2-xy)\hat{k}]\cdot\hat{k} = c^2 - xy$

$$\therefore \iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a (c^2-xy)\,dx\,dy = \int_0^b \left(c^2a - \frac{a^2}{2}y\right)dy = abc^2 - \frac{a^2b^2}{4}$$

On S_3 $(x=0)$, we have $\hat{n} = -\hat{i}$, $\vec{F} = -yz\hat{i} + y^2\hat{j} + z^2\hat{k}$

So that $\vec{F}\cdot\hat{n} = (-yz\hat{i} + y^2\hat{j} + z^2\hat{k})\cdot(-\hat{i}) = yz$

$$\therefore \iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b yz\,dy\,dz = \int_0^c \frac{b^2}{2}x\,dz = \frac{b^2c^2}{4}$$

On S_4 $(x=a)$, we have $\hat{n} = \hat{i}$, $\vec{F} = (a^2-yz)\hat{i} + (y^2-az)\hat{j} + (z^2-ay)\hat{k}$

So that $\vec{F}\cdot\hat{n} = [(a^2-yz)\hat{i} + (y^2-az)\hat{j} + (z^2-ay)\hat{k}]\cdot\hat{i}$

$$= a^2 - yz$$

$$\therefore \iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b (a^2-yz)\,dy\,dz = \int_0^c \left(a^2b - \frac{b^2}{2}z\right)dz$$

$$= a^2bc - \frac{b^2c^2}{4}$$

On S_5 $(y=0)$, we have $\hat{n} = -\hat{j}$, $\vec{F} = x^2\hat{i} - zx\hat{j} + z^2\hat{k}$

So that $\vec{F}\cdot\hat{n} = (x^2\hat{i} - zx\hat{j} + z^2\hat{k})\cdot(-\hat{j}) = zx$

$$\therefore \iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^c zx\,dz\,dx = \int_0^a \frac{c^2}{2}x\,dx = \frac{a^2c^2}{2}$$

On S_6 $(y=b)$, we have $\hat{n} = \hat{j}$, $\vec{F} = (x^2-bz)\hat{i} + (b^2-zx)\hat{j} + (z^2-bx)\hat{k}$

So that $\vec{F}\cdot\hat{n} = [(x^2-bz)\hat{i} + (b^2-zx)\hat{j} + (z^2-bx)\hat{k}]\cdot\hat{j} = b$

$$\therefore \iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^c (b^2-zx)\,dz\,dx = \int_0^a \left(b^2c - \frac{c^2}{2}x\right)dx = ab^2c$$

$-$

$$\therefore \int_S \vec{F}\cdot\hat{n}\,ds = \frac{a^2b^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + \frac{a^2c^2}{4} + ab$$

$-$

$$= abc(a+b+c) \quad \text{Verified}$$

Q.2: Verify Gauss Divergence theorem for $\int_C [(x^3-yz)\hat{i} - 2x^2y\hat{j} + z\hat{k}] \cdot \hat{n}\, ds$ where S denotes the surface of cube bounded by the planes $x=0$, $x=a$; $y=0$, $y=a$; $z=0$, $z=a$. (2016-17)

Sol: Proceed as Q.1.

Q.3: Verify the divergence theorem for $\vec{F} = (x^3-yz)\hat{i} + (y^3-zx)\hat{j} + (z^3-xy)\hat{k}$, taken over the cube bounded by planes $x=0, y=0, z=0, x=1, y=1, z=1$. (2018-19)

Sol: Proceed as Q.1.

Q.4: Verify the divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz$, taken over the rectangular parallelepiped $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$. (2019-20)

Sol: Proceed as Q.1.

Q.5: Use divergence theorem to evaluate the surface integral $\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ where S is the portion of the plane $x+2y+3z=6$ which lies in the first octant. (2020-21)

Sol: By Gauss' Divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V \operatorname{div}\vec{F}\, dV \quad \text{——} ①$$

Here $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$, $\hat{n}\,ds = dy\,dz\,\hat{i} + dz\,dx\,\hat{j} + dx\,dy\,\hat{k}$

Now

$$\operatorname{div}\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

---

from (1),

$$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy) = \iiint_V 3\, dv$$

$$= 3\int_{x=0}^{6}\int_{y=0}^{\frac{6-x}{2}}\int_{z=0}^{\frac{6-x-2y}{3}} dz\,dy\,dx$$

$$= 3\int_{x=0}^{6}\int_{y=0}^{\frac{6-x}{2}} \left(\frac{6-x-2y}{3}\right) dy\,dx$$

$$= \int_{x=0}^{6} \{(6-x)y - y^2\Big]_0^{\frac{6-x}{2}} dx$$

$$= \int_{x=0}^{6} \left[\frac{(6-x)^2}{2} - \frac{(6-x)^2}{4}\right] dx$$

$$= \int_{x=0}^{6} \frac{(6-x)^2}{4}\, dx$$

$$= \frac{1}{4}\left[\frac{(6-x)^3}{(-3)}\right]_0^6 = \frac{1}{12}(216) = 18.$$

Ques: Find $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$ and S is the surface of the sphere having centre at $(3,-1,2)$ and radius 3.

Sol: By Gauss divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dv$$

Now $\operatorname{div}\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left\{(2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}\right\}$

$= 2-1+2 = 3$

So $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV$

$= \iiint_V 3\,dV$

$= 3V$

But V is the volume of a sphere of radius 3

$\therefore \quad V = \frac{4}{3}\pi(3)^3 = 36\pi$.

Hence $\iint_S \vec{F}\cdot\hat{n}\,ds = 3\times 36\pi = 108\pi$

5 Years AKTU University Examination Questions		Unit-5	
S.No	Questions	Session	Lecture No
1	Find a unit vector normal to the surface $x^3+y^3+3xyz=3$ at the point (1,2,-1).	2013-14 (Very short)	33-39
2	If $u=x+y+z$, $v=x^2+y^2+z^2$, $w=yz+zx+xy$. Prove that $grad\,u$, $grad\,v$ and $grad\,w$ are coplanar.	2014-15 (Long)	33-39
3	For the scalar field $u=\frac{x^2}{2}+\frac{y^2}{3}$, find the magnitude of gradient at the point (1,3).	2016-17 (Very short)	33-39
4	Define det ∇ operator and gradient.	2018-19 (Very short)	33-39
5	If $\phi=3x^2y-y^3z^2$, find $grad\phi$ at point (2, 0, -2).	2018-19 (Very short)	33-39
6	Find $grad\,\phi$ at the point (2,1,3) where $\phi=x^2+yz$	2019-20 (Very short)	33-39
7	Find the directional derivative of v^2, where $\vec{v}=xy^2\hat{i}+zy^2\hat{j}+xz^2\hat{k}$ at the point $(2,0,3)$ in the direction of the outward normal to the sphere $x^2+y^2+z^2=14$ at the point $(3,2,1)$.	2012-13 (Short)	33-39
8	Find the directional derivative of: $\phi=(x^2+y^2+z^2)^{-1/2}$ at the point (3,1,2) in the direction of the vector $yz\hat{i}+zx\hat{j}+xy\hat{k}$.	2013-14 (Short)	33-39
9	Find the directional derivative of $\left(\frac{1}{r^2}\right)$ in the direction of $\vec{r}$, where $\vec{r}=\hat{i}x+\hat{j}y+\hat{k}z$.	2016-17 (Short)	33-39
10	Find the directional derivative of $\phi=5x^2y-5y^2z+\frac{5}{2}z^2x$ at the point P(1,1,1) in the direction of the line $\frac{x-1}{2}=\frac{y-3}{-2}=\frac{z}{1}$.	2018-19 (Long)	33-39
11	Find the directional derivative of $\phi(x,y,z)=x^2yz+4xz^2$ at (1,-2,1) in the direction of $2\hat{i}-\hat{j}-2\hat{k}$. Find also the greatest rate of increase of ϕ.	2019-20 (Long)	33-39
12	Show that the vector: $\vec{V}=3y^4z^2\hat{i}+4x^3z^2\hat{j}-3x^2y^2\hat{k}$ is solenoidal.	2011-12 (Very short)	33-39
13	Find the value of m if $\vec{F}=mx\hat{i}-5y\hat{j}+2z\hat{k}$ is a solenoidal vector.	2017-18 (Very short)	33-39
14	Show that vector $\vec{V}=(x+3y)\hat{i}+(y-3z)\hat{j}+(x-2z)\hat{k}$ is solenoidal.	2019-20, 2020-21 (Very short)	33-39
15	If $\vec{F}=(\vec{a}\cdot\vec{r})\vec{r}$, where $\vec{a}$ is a constant vector, find $curl\,\vec{F}$ and prove that it is perpendicular to $\vec{a}$.	2011-12 (Short)	33-39
16	If $\vec{F}=\frac{\vec{r}}{r^3}$, find $curl\,\vec{F}$.	2012-13 (Very short)	33-39
17	Prove that $\vec{A}=(6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}$ is irrotational.	2013-14 (Very short)	33-39
18	Find the curl of $\vec{F}=xy\hat{i}-y^2\hat{j}+xz\hat{k}$ at (-2,4,1).	2015-16 (Very short)	33-39
19	Prove that, for every field $\vec{V}$; $div\,curl\,\vec{V}=0$.	2015-16 (Short)	33-39
20	A fluid motion is given by $\vec{v}=(y+z)\hat{i}+(z+x)\hat{j}+(x+y)\hat{k}$. Show that the motion is irrotational and hence find the velocity potential.	2015-16 (Short)	33-39
21	If $\vec{A}=(xz^2\hat{i}+2y\hat{j}-3xz\hat{k})$ and $B=(3xz\hat{i}+2yz\hat{j}-z^2\hat{k})$. Find the value of $[\vec{A}\times(\nabla\times\vec{B})]$ & $[(\vec{A}\times\nabla)\times\vec{B}]$.	2016-17 (Short)	33-39
22	Determine the value of constants a,b,c if $\vec{F}=(x+2y+az)\hat{i}+(bx-3y-z)\hat{j}+(4x+cy+2z)\hat{k}$ is irrotational.	2017-18 (Short)	33-39
23	If all second order derivatives of ϕ and $\vec{v}$ are continuous, then show that (i) $Curl(grad\,\phi)=0$ (ii) $div(curl\,\vec{v})=0$	2017-18 (Long)	33-39
24	Prove that $(y^2-z^2+3yz-2x)\hat{i}+(3xz+2xy)\hat{j}+(3xy-2xz+2z)\hat{k}$ is both solenoidal and irrotational.	2018-19 (Long)	33-39
25	A fluid motion is given by $\vec{v}=(y\sin z-\sin x)\hat{i}+(x\sin z+2yz)\hat{j}+(xy\cos z+y^2)\hat{k}$. Is the motion irrotational? If so, find the velocity potential.	2020-21 (Long)	33-39
26	Evaluate: $\iint_S \vec{F}\cdot\hat{n}\,dS$ where $\vec{F}=18z\hat{i}-12\hat{j}+3y\hat{k}$ and S is the part of the plane $2x+3y+6z=12$ in the first octant.	2011-12 (Long)	33-39
27	Find the work done in moving a particle in the force field: $\vec{F}=3x^2\hat{i}+(2xz-y)\hat{j}+z\hat{k}$ along the curve $x^2=4y$ and $3x^3=8z$ from $x=0$ to $x=2$.	2011-12 (Short)	33-39
28	Evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the curve $x^2+y^2=1$, $z=1$ in the positive direction from $(0,1,1)$ to $(1,0,1)$, where: $\vec{F}=(yz+2x)\hat{i}+xz\hat{j}+(xy+2z)\hat{k}$.	2012-13 (Short)	33-39
29	If $\vec{A}=(x-y)\hat{i}+(x+y)\hat{j}$, evaluate $\oint_C \vec{A}\cdot d\vec{r}$ around the curve C consisting of $y=x^2$ and $y^2=x$.	2013-14 (Short)	33-39
30	Evaluate $\int_S (yzI+zxJ+xyK)\cdot dS$, where S is the surface of the sphere $x^2+y^2+z^2=a^2$ in the first octant.	2014-15 (Long)	33-39
31	If $\vec{A}=(x-y)\hat{i}+(x+y)\hat{j}$, evaluate $\oint_C \vec{A}\cdot d\vec{r}$ around the curve C consisting of $y=x^2$ and $y^2=x$.	2017-18 (Short)	33-39

32	State Green's theorem for a plane region.	2011-12 (Very short)	33-39
33	Using Green's theorem, evaluate the integral $\oint_C (xy\,dy - y^2 dx)$, where C is the square cut-from the first quadrant by the lines $x = 1, y = 1$.	2012-13 (Very short)	33-39
34	Verify Green's theorem in plane for: $\oint_C (x^2 - 2xy)dx + (x^2 y + 3)dy$, where C is the boundary of the region defined by $y^2 = 8x$ and $x = 2$.	2013-14 (Long)	33-39
35	Verify the Green's theorem to evaluate the line integral $\int_C (2y^2 dx + 3xdy)$, where C is the boundary of the closed region bounded by $y = x$ and $y = x^2$.	2015-16 (Long)	33-39
36	If $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$, then evaluate the value of $\oint \vec{F} \cdot dr$.	2016-17 (Short)	33-39
37	Verify Green's theorem, evaluate $\int_C (x^2 + xy)dx + (x^2 + y^2)dy$ where C square formed by lines $x = \pm 1, y = \pm 1$.	2017-18 (Long)	33-39
38	State Green's theorem.	2020-21 (Very short)	33-39
39	Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ by Stoke's Theorem, where: $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of triangle with vertices at (0,0,0), (1,0,0) and (1,1,0).	2013-14 (Short)	33-39
40	Verify Stokes theorem for $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0$ and $y = b$.	2014-15 (Long)	33-39
41	State Stoke's theorem.	2015-16 (Very short)	33-39
42	Verify Stokes theorem $\vec{F} = (2y + z, x - z, y - x)$ taken over the triangle ABC cut from the plane $x + y + z = 1$ by the coordinate planes.	2016-17 (Short)	33-39
43	Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$ taken round the rectangle bounded by the lines $x = \pm a, y = 0, y = b$.	2017-18 (Long)	33-39
44	Verify Stoke's theorem for the vector field $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ integrated round the rectangle in the plane $z = 0$ and bounded by the lines $x = 0, y = 0, x = a, y = b$.	2019-20 (Short)	33-39
45	Verify Stoke's theorem for the function $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated round the square whose sides are $x = 0, y = 0, x = a, y = a$ in the plane $z = 0$.	2020-21 (Long)	33-39
46	Verify the Gauss divergence theorem for: $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}$ Taken over the rectangular parallelepiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.	2012-13 (Long)	33-39
47	Verify Gauss Divergence theorem for $\int_C [(x^3 - yz)\hat{i} - 2x^2 y\hat{j} + 2\hat{k}]\hat{n}\, dS$, where S denotes the surface of cube bounded by the planes $x = 0, x = a, y = 0, y = a, z = 0, z = a$.	2016-17 (Short)	33-39
48	Verify the divergence theorem for $\vec{F} = (x^3 - yz)\hat{i} + (y^3 - zx)\hat{j} + (z^3 - xy)\hat{k}$, taken over the cube bounded by planes $x = 0, y = 0, z = 0, x = 1, y = 1, z = 1$.	2018-19 (Very short)	3
49	Verify the divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ taken over the rectangular parallelepiped $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.	2019-20 (Long)	3
50	Use divergence theorem to evaluate the surface integral $\iint_S (xdydz + ydzdx + zdxdy)$ where S is the portion of the plane $x + 2y + 3z = 6$ which lies in the first octant.	2020-21 (Long)	3